

Classical and quantum scattering by a gravitational center

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July 24, 2008

Abstract

The small angle scattering (by a gravitational field) of classical and quantum particles is considered and compared. It is suggested that the differences in small angle scattering of particles with spin 0, 1, 2 are due to the nonzero probability of forward scattering for particles described by a wave packet. It is suggested that measurements of the deflection of light in the vicinity of the Sun will decide which coordinate system is the privileged one.

1 Introduction

When a photon is deflected by the Sun, we can know with good accuracy its impact parameter ρ , its frequency and the coordinates of place where it is observed. We note that we always observe at a finite distance from the Sun and in principle the place of observation is at our disposal. Classical theory permits to predict the results of such observations. So we expect that the classical approach should be applicable in rather wide range of impact parameters and values of $r_g = 2Gm/c^2$. The classical cross section may also serve as a guide for quantum calculation beyond the Born approximation. To some extent this should be also true in Coulomb scattering [1].

For a scattering with the momentum transfer q the impact parameter ρ is of order of \hbar/q and the formation length of this process is of order of several \hbar/q . Beyond this length the particle is unable to obtain the momentum transfer q . In quantum theory in Born approximation the particle obtains the required momentum transfer by interacting with only one graviton. In principle the quantum particle can pass the formation length without interaction or interacting with several gravitons in such a way that no momentum transfer is passed to it. It is very interesting to know the probability for this process. If it is nonzero, then the concept of curved space time is of only limited validity. Due to the possibility of flying by without deflection the quantum cross section should be smaller then the classical one. So the detailed study of classical and quantum scattering is of interest in many respects.

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2 Small angle classical scattering by the Schwarzschild field

A) Particle with mass.

The trajectories in the Schwarzschild field were studied in a number of papers [2-4], see also [5]. If the standard coordinate system

$$ds^2 = \left(1 - \frac{r_g}{r}\right) (dx^0)^2 - r^2(\sin^2 \theta d\varphi^2 + d\theta^2) - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 \quad (1)$$

is used, the equation governing the scattering trajectory has the simplest form

$$\frac{du}{d\vartheta} = \pm \sqrt{f(u)}, \quad u = \frac{\rho}{r}, \quad \delta = \frac{r_g}{\rho}, \quad r_g = \frac{2GM}{c^2},$$

$$f(u) = 1 - u^2 + \varkappa u \delta + u^3 \delta = \delta(u - u_1)(u - u_2)(u - u_3), \quad \varkappa = \beta^{-2} - 1. \quad (2)$$

Here β is velocity at infinity in units of c , the signs $+$ and $-$ in front of the square root refer to the first and the second halves of trajectory respectively. From (2) the half angle between the asymptotes is

$$\vartheta_{1/2} = \int_0^{\vartheta_{1/2}} d\vartheta = \int_0^{u_2} \frac{du}{\sqrt{f(u)}}. \quad (3)$$

This is the contribution from the first part of the trajectory. It is assumed that $u_1 < u_2 < u_3$. Similarly, for the second half:

$$2\vartheta_{1/2} - \vartheta_{1/2} = \int_{\vartheta_{1/2}}^{2\vartheta_{1/2}} d\vartheta = - \int_{u_2}^0 \frac{du}{\sqrt{f(u)}} = \int_0^{u_2} \frac{du}{\sqrt{f(u)}} = \vartheta_{1/2}. \quad (4)$$

So the angle between asymptotes is $2\vartheta_{1/2}$. The scattering angle is $\theta = 2\vartheta_{1/2} - \pi$. Each half of the trajectory contributes $\vartheta_{1/2}$.

Now the expressions for roots u_1, u_2, u_3 as functions of δ and \varkappa can be obtained by the method of Newton: if $x^{(0)}$ is the zero order approximation for the root of $f(x) = 0$, then in the first approximation we have $x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})}$. In the second approximation $x^{(2)} = x^{(1)} - \frac{f(x^{(1)})}{f'(x^{(1)})}$ and so on. Thus, for the root u_2 starting from the zero order approximation $u_2^{(0)} = 1$, we have for the function $f(u)$ in (2) $f(1) = (\varkappa + 1)\delta$. From $f'(u) = -2u + \varkappa\delta + 3u^2\delta$ we get $f'(1) = -2$. So $u_2^{(1)} = 1 + \frac{1}{2}(1 + \varkappa)\delta$. In the second approximation $f(u_2^{(1)}) = (\frac{5}{2^2} + \frac{3\varkappa}{2} + \frac{\varkappa^2}{2^2})\delta$. As for $f'(u_2^{(1)})$ we may in the considered approximation still take $f'(1) = -2$. So we get $u_2^{(2)} = 1 + \frac{1}{2}(1 + \varkappa)\delta + (\frac{5}{2^3} + \frac{3\varkappa}{2^2} + \frac{\varkappa^2}{2^3})\delta^2$. Continuing this process, we find

$$u_2 = 1 + \frac{1}{2}(1 + \varkappa)\delta + (\frac{5}{2^3} + \frac{3\varkappa}{2^2} + \frac{\varkappa^2}{2^3})\delta^2 + (1 + \frac{3\varkappa}{2} + \frac{\varkappa^2}{2})\delta^3 + \dots \quad (5)$$

Now the root u_1 can be obtained from here if we note that according to (2) $f(u) \equiv f(u, \delta) = f(-u, -\delta)$. Hence,

$$u_1 \equiv u_1(\delta) = -u_2(-\delta) = -1 + \frac{1}{2}(1 + \varkappa)\delta - (\frac{5}{2^3} + \frac{3\varkappa}{2^2} + \frac{\varkappa^2}{2^3})\delta^2 + (1 + \frac{3\varkappa}{2} + \frac{\varkappa^2}{2})\delta^3 + \dots \quad (6)$$

The expansion for u_3 can be obtained in the same way as for u_2

$$u_3 = \frac{1}{\delta} - (1 + \varkappa)\delta - (2 + 3\varkappa + \varkappa^2)\delta^3 + \dots \quad (7)$$

It is easy to check that $u_1 + u_2 + u_3 = \delta^{-1}$ as it should be.

From (2) and (3) we have in agreement with [2-4]

$$\begin{aligned} \vartheta_{1/2} &= \frac{1}{\sqrt{\delta}} \int_0^{u_2} \frac{du}{\sqrt{(u - u_3)(u - u_2)(u - u_1)}} = \\ &= \frac{2}{\sqrt{(u_3 - u_1)\delta}} F(\phi, \kappa), \quad \kappa = \left(\frac{u_2 - u_1}{u_3 - u_1} \right)^{1/2}, \quad \sin^2 \phi = \frac{1 - u_1 u_3^{-1}}{1 - u_1 u_2^{-1}}. \end{aligned} \quad (8)$$

Here $F(\phi, \kappa)$ is the elliptical integral. Using (5), (6) and (7) we get, retaining terms up to δ^2

$$\kappa^2 = 2\delta(1 - \delta + \dots), \quad \sin^2 \phi = \frac{1}{2} \left\{ 1 + \frac{3 + \varkappa}{2} \delta + O(\delta^3) \right\}. \quad (9)$$

From here

$$\sin \phi = \frac{1}{\sqrt{2}} \left[1 + \frac{3 + \varkappa}{2^2} \delta - \frac{(3 + \varkappa)^2}{2^5} \delta^2 + \dots \right], \quad \cos \phi = \frac{1}{\sqrt{2}} \left[1 - \frac{3 + \varkappa}{2^2} \delta + \dots \right]. \quad (10)$$

To get ϕ from the expression for $\sin^2 \phi$ it is worth-while to make the substitution $\phi = \frac{\pi}{4} + \psi$, $\sin^2 \phi = \frac{1}{2} + \frac{1}{2} \sin 2\psi$. (This is especially useful when more terms are needed than we retain here, see the next subsection.) So for ψ we have

$$\sin 2\psi = \frac{3 + \varkappa}{2} \delta + O(\delta^3). \quad (11)$$

So $2\psi = \frac{3 + \varkappa}{2} \delta + O(\delta^3)$. Hence,

$$\phi = \frac{\pi}{4} + \psi = \frac{\pi}{4} + \frac{3 + \varkappa}{2^2} \delta + O(\delta^3). \quad (12)$$

As δ is assumed to be small, κ^2 is small and we can use the expansion, see equation (5) in section (13.6) in [6]

$$F(\phi, \kappa) = S_0 + \frac{1}{2} S_2 \kappa^2 + \frac{3}{2^3} S_4 \kappa^5 + \dots, \quad S_{2n} \equiv S_{2n}(\phi) = \int_0^\phi (\sin t)^{2n} dt; \quad (13)$$

$$S_2 = \frac{1}{2}(\phi - \sin \phi \cos \phi); \quad S_4 = \frac{3}{8}\phi - \frac{3}{8} \sin \phi \cos \phi - \frac{1}{4} \sin^3 \phi \cos \phi. \quad (14)$$

From (6) and (7) we have

$$(u_3 - u_1)\delta = 1 + \delta - \frac{3}{2}(1 + \varkappa)\delta^2 + \dots. \quad (15)$$

Using equation (3.6.19) in [7] we obtain from here

$$[(u_3 - u_1)\delta]^{-1/2} = 1 - \frac{1}{2}\delta + \left(\frac{3}{8} + \frac{\varkappa}{4} \right) 3\delta^2 + \dots. \quad (16)$$

Performing remaining calculation, we get for $\vartheta_{1/2}$ in (8)

$$\vartheta_{1/2} = \frac{\pi}{2} + (1 + \frac{\varkappa}{2})\delta + \pi \left(\frac{15}{32} + \frac{3\varkappa}{8} \right) \delta^2 + \dots \quad (17)$$

The scattering angle is $\theta = 2\vartheta_{1/2} - \pi$. So

$$\frac{\theta}{2} = \vartheta_{1/2} - \frac{\pi}{2} = (1 + \frac{\varkappa}{2})\delta + \pi \left(\frac{15}{32} + \frac{3\varkappa}{8} \right) \delta^2 + \dots \quad (18)$$

Using equation (3.6.25) in [7] we find

$$\delta = \frac{\theta}{2 + \varkappa} - \frac{5 + 4\varkappa}{(2 + \varkappa)^3} \frac{3\pi}{2^4} \theta^2 + \dots \quad (19)$$

From here with the help of (3.6.25) in [7] we find

$$\delta^{-2} = \left(1 + \frac{\varkappa}{2}\right)^2 \left(\frac{2}{\theta}\right)^2 \left\{1 + \frac{5 + 4\varkappa}{(2 + \varkappa)^2} \frac{3\pi}{8} \theta + \dots\right\}. \quad (20)$$

This gives the classical integral cross section

$$\sigma_{cl}(\theta) = \pi \rho^2(\theta), \quad \delta^{-2} \equiv \rho^2/r_g^2. \quad (21)$$

If we compare (18) with correspondind equation (10.14) in [4] we find that only the leading term is the same. The preceding equation (10.13) in [4] contains errors in powers of the root of the cubic.

The differential cross section is obtained from (21) by differentiation, multiplying by π and changing the averall sign in the right hand side:

$$d\sigma_{cl}(\theta) = 2\pi r_g^2 (1 + \frac{1}{2}\varkappa)^2 \frac{1}{y^3} \{1 + \frac{5 + 4\varkappa}{(2 + \varkappa)^2} \frac{3\pi}{8} y + \dots\} dy, \quad y = \frac{\theta}{2}. \quad (21a)$$

In quantum picture the first Born approximation for scalar particle cross section is given in equation (39) below. It contains only even powers of y as in Coulomb scattering [1]. This should mean that corrections to the first Born approximation must contain classical terms independent of \hbar . In Coulomb scattering these classical terms in higher approximations may be expected only at unrealistically high $\alpha Z \gg 1$.

B) Massless particle

In this subsection we obtain more terms of the expansions for the case of massless particle i.e. for $\varkappa = 0$, see (2). Continuing the process described in getting (5) we find

$$u_2 = 1 + \frac{1}{2}\delta + \frac{5}{2^3}\delta^2 + \delta^3 + \frac{3 \cdot 7 \cdot 11}{2^7}\delta^4 + \frac{7}{2}\delta^5 + \dots, \quad (22)$$

$$u_3 = \frac{1}{\delta} - \delta - 2\delta^3 - 7\delta^5 + \dots, \quad (23)$$

$$u_1 \equiv u_1(\delta) = -u_2(-\delta) = -1 + \frac{1}{2}\delta - \frac{5}{2^3}\delta^2 + \delta^3 - \frac{231}{2^7}\delta^4 + \frac{7}{2}\delta^5 + \dots \quad (24)$$

Using these expressions we find for κ^2 and $\sin^2 \phi$, see (7)

$$\kappa^2 = 2\delta\{1 - \delta + \frac{5^2}{2^3}\delta^2 - \frac{3 \cdot 7}{2^2}\delta^3 - \frac{7 \cdot 281}{2^7}\delta^4 + \dots\}, \quad (25)$$

$$\sin^2 \phi = \frac{1}{2}\{1 + \frac{3}{2}\delta + \frac{3 \cdot 11}{2^4}\delta^3 + \frac{3^2 \cdot 5 \cdot 37}{2^8}\delta^5 + \dots\}. \quad (26)$$

From here

$$\sin \phi = \frac{1}{\sqrt{2}}\{1 + \frac{3}{4}\delta - \frac{3^2}{2^5}\delta^2 + \frac{3 \cdot 53}{2^7}\delta^3 - \frac{3^2 \cdot 221}{2^{11}}\delta^4 + \frac{3^2 \cdot 3941}{2^{13}}\delta^5 + \dots\}. \quad (27)$$

The term with δ^5 is needed only if we want to obtain ϕ from (27). But we proceed as in obtaining (11):

$$\sin 2\psi = \frac{3}{2}\delta\{1 + \frac{11}{2^3}\delta^2 + \frac{3 \cdot 5 \cdot 37}{2^7}\delta^4 + \dots\}. \quad (28)$$

From here

$$2\psi = \frac{3}{2}\delta + \frac{3 \cdot 7}{2^3}\delta^3 + \frac{3^2 \cdot 167}{2^5 \cdot 5}\delta^5 + \dots. \quad (29)$$

Hence

$$\phi = \frac{\pi}{4} + \psi = \frac{\pi}{4} + \frac{3}{2}\delta + \frac{3 \cdot 7}{2^4}\delta^3 + \frac{3^2 \cdot 167}{2^6 \cdot 5}\delta^5 + \dots. \quad (30)$$

From (27) we obtain

$$\cos \phi = \frac{1}{\sqrt{2}}\{1 - \frac{3}{4}\delta - \frac{3^2}{2^5}\delta^2 - \frac{3 \cdot 53}{2^7}\delta^3 - \frac{3^2 \cdot 221}{2^{11}}\delta^4 + \dots\}. \quad (31)$$

In equation (16) (with $\varkappa = 0$) we have now more terms:

$$[(u_3 - u_1)\delta]^{-1/2} = 1 - \frac{1}{2}\delta + \frac{3^2}{2^3}\delta^2 - \frac{7}{2^2}\delta^3 + \frac{5^2 \cdot 23}{2^7}\delta^4 - \frac{3^3 \cdot 5}{2^4}\delta^5 + \dots. \quad (32)$$

Using expressions for $\sin \phi$ and $\cos \phi$ in (27) and (31) we obtain S_{2n} which are polynomials in $\sin \phi$ and $\cos \phi$. Then using also (25) we find similarly to (13)

$$F(\phi, \kappa) = S_0 + \frac{1}{2}S_2\kappa^2 + \frac{3}{2^3}S_4\kappa^4 + \frac{5}{2^4}S_6\kappa^6 + \frac{5 \cdot 7}{2^7}S_8\kappa^8 + \frac{3^2 \cdot 7}{2^8}S_{10}\kappa^{10} + \dots = \frac{\pi}{4} + \left(\frac{\pi}{2^3} + \frac{1}{2}\right)\delta \quad (33)$$

$$+ \left(\frac{\pi}{2^6} + \frac{1}{2^2}\right)\delta^2 + \left(\frac{39\pi}{2^7} + \frac{43}{2^4 \cdot 3}\right)\delta^3 + \left(\frac{313\pi}{2^{12}} + \frac{5^2}{2^3 \cdot 3}\right)\delta^4 + \left(\frac{7 \cdot 1487\pi}{2^{13}} + \frac{12689}{2^8 \cdot 3 \cdot 5}\right)\delta^5. \quad (34)$$

Using (32) we get from (8) and (34)

$$\vartheta_{1/2} = \frac{\pi}{2} + \delta + \frac{3 \cdot 5\pi}{2^5}\delta^2 + \frac{2^3}{3}\delta^3 + \frac{3^2 \cdot 5 \cdot 7 \cdot 11\pi}{2^{11}}\delta^4 + \frac{2^3 \cdot 7}{5}\delta^5 \dots, \quad (35)$$

and similarly to (19) we obtain

$$\delta = y\{1 - \frac{3 \cdot 5\pi}{2^5}y + \left(\frac{3^2 \cdot 5^2\pi^2}{2^9} - \frac{2^3}{3}\right)y^2 + \left(\frac{5 \cdot 1867\pi}{2^{11}} - \frac{3^3 \cdot 5^4\pi^3}{2^{15}}\right)y^3 +$$

$$\left(\frac{2^3 \cdot 19}{3 \cdot 5} - \frac{3^2 \cdot 5^2 \cdot 7 \cdot 157 \pi^2}{2^{15}} + \frac{3^4 \cdot 5^4 \cdot 7 \pi^4}{2^{19}} \right) y^4 + \dots \}, \quad y = \frac{\theta}{2} = \vartheta_{1/2} - \frac{\pi}{2}. \quad (36)$$

As in (20) we find

$$\delta^{-2} \equiv \frac{\rho^2}{r_g^2} = y^{-2} \{1 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4 + \dots\}, \quad (37)$$

$$\begin{aligned} c_1 &= \frac{3 \cdot 5 \pi}{2^4}, \quad c_2 = \frac{2^4}{3} - \frac{3^2 \cdot 5^2 \pi^2}{2^{10}}, \\ c_3 &= -\frac{5 \cdot 331 \pi}{2^{10}} + \frac{3^3 \cdot 5^3 \pi^3}{2^{14}}, \quad c_4 = \frac{2^4}{3 \cdot 5} + \frac{3^2 \cdot 5^2 \cdot 331 \pi^2}{2^{15}} - \frac{3^4 \cdot 5^5 \pi^4}{2^{20}}. \end{aligned} \quad (37a)$$

The approximate value for $c_i, i = 1, 2, 3, 4$ are

$$c_1 = 2.945, \quad c_2 = 3.165, \quad c_3 = 1.310, \quad c_4 = -0.016. \quad (37b)$$

The differential cross section is obtained from (37) by differentiation

$$d\sigma(\theta)_{cl} = \pi r_g^2 \frac{2^3}{\theta^3} \{1 + c_1 \frac{\theta}{2^2} - c_3 \frac{\theta^3}{2^4} - c_4 \frac{\theta^4}{2^4} + \dots\} d\theta. \quad (38)$$

It is interesting to compare this classical cross section with the quantum one. For massless scalar particle we have in the first Born approximation [8]

$$d\sigma_{sc} = 2\pi r_g^2 (1 + \frac{\varkappa}{2})^2 \frac{\cos y}{\sin^3 y} dy = 2\pi r_g^2 (1 + \frac{\varkappa}{2})^2 \frac{1}{y^3} \{1 - \frac{y^4}{15} + \dots\} dy, \quad y = \frac{\theta}{2}. \quad (39)$$

The substitution $y \rightarrow -y$ changes the sign of $\sin y$ and y . Hence the expression in braces in (39) can contain only even powers of y . We note also that c_4 in (38) is negative ($c_4 = -0.016$) in contrast with corresponding coefficient $1/15$ in (39).

The quantum cross section for photon contains an additional factor:

$$d\sigma_\gamma = d\sigma_{sc} \cos^4 y, \quad (40)$$

see[8] and references therein. The factor $d\sigma_{sc}$ in (40) and below is taken at $\varkappa = 0$ i.e. for massless particle. So for $y > 0$ we have

$$d\sigma_\gamma < d\sigma_{sc}$$

For graviton [9], [10]:

$$d\sigma_g = d\sigma_{sc} \frac{1}{8} (1 + 6 \cos^2 \theta + \cos^4 \theta) = d\sigma_{sc} (\cos^8 y + \sin^8 y). \quad (40a)$$

Hence $d\sigma_g < d\sigma_\gamma$ for $0 < y << 1$. So for small angle scattering the cross section is smaller for particle with higher spin. It seems reasonable to expect from these facts that spin facilitates forward scattering of a particle described by a wave packet.

3 Small angle classical scattering by the linear approximation of the Schwarzschild field

To see the effects of the nonlinearity of the Schwarzschild field on scattering, we consider in this Section the linear approximation of the isotropic coordinates. This corresponds to the quantum treatment of scattering in [8-10]. Only massless particle is considered here.

Proceeding in the same way as in [3] and [4] we obtain instead of (2)

$$\frac{du}{d\vartheta} = \pm \sqrt{\frac{f(u)}{1-u\delta}}, \quad f(u) = 1 - u^2 + (u^3 + u)\delta = \delta(u - u_1)(u - u_2)(u - u_3). \quad (41)$$

So we use in this Section the same notation as before, but with a somewhat different meaning. For $\vartheta_{1/2}$ we get a more complicated expression:

$$\vartheta_{1/2} = \frac{1}{\sqrt{\delta}} \int_0^{u_2} \sqrt{\frac{1-u\delta}{(u_3-u)(u_2-u)(u-u_1)}} du = \frac{1}{\sqrt{u_3\delta}} \int_0^{u_2} \sqrt{\frac{1-u\delta}{(1-uu_3^{-1})}} \frac{du}{\sqrt{R(u)}}. \quad (42)$$

Here $R(u) = (u_2 - u)(u - u_1)$.

From (41) we find instead of (22-24)

$$u_2 = 1 + \delta + \frac{3}{2}\delta^2 + 3\delta^3 + \frac{5 \cdot 11}{2^3}\delta^4 + 17\delta^5 + \dots, \quad (43)$$

$$u_3 = \frac{1}{\delta} - 2\delta - 6\delta^3 - 34\delta^5 + \dots, \quad (44)$$

$$u_1 \equiv u_1(\delta) = -u_2(-\delta) = -1 + \delta - \frac{3}{2}\delta^2 + 3\delta^3 - \frac{5 \cdot 11}{2^3}\delta^4 + 17\delta^5 + \dots. \quad (45)$$

For $\delta \ll 1$ we have $u_3 \gg 1$, Hence $u/u_3 \ll 1$ in the r.h.s. of (42). So to evaluate (42) we can proceed as follows

$$\begin{aligned} \frac{1-u\delta}{1-uu_3^{-1}} &= \frac{(1-u\delta - 2u\delta^3 - 10u\delta^5) + 2u\delta^3(1+5\delta^2)}{1-u\delta(1+2\delta^2+10\delta^4\dots)} = \\ 1 + \frac{2u\delta^3(1+5\delta^2)}{1-u\delta-2u\delta^3+\dots} &= 1 + 2u\delta^3[1+u\delta+(u^2+5)\delta^2+\dots]. \end{aligned} \quad (46)$$

From here

$$\left(\frac{1-u\delta}{1-uu_3^{-1}} \right)^{1/2} = 1 + u(\delta^3 + 5\delta^5) + u^2\delta^2 + u^3\delta^5 + \dots. \quad (47)$$

Using this in (42) we have

$$\vartheta_{1/2} = \frac{1}{\sqrt{u_3\delta}} \{I_0 + (\delta^3 + 5\delta^5)I_1 + \delta^4 I_2 + \delta^5 I_3 + \dots\}, \quad I_n = \int_0^{u_2} \frac{u^n du}{\sqrt{R(u)}}. \quad (48)$$

Evaluating these integrals we find

$$I_0 = \frac{\pi}{2} + \arcsin \frac{u_2 + u_1}{u_2 - u_1}, \quad I_1 = \sqrt{-u_2 u_1} + \frac{u_1 + u_2}{2} I_0,$$

$$I_2 = 3 \frac{u_1 + u_2}{4} \sqrt{-u_1 u_2} + \left\{ \frac{3(u_1 + u_2)^2}{8} - \frac{u_1 u_2}{2} \right\} I_0, \quad R(0) = -u_1 u_2, \quad R(u_2) = 0. \quad (49)$$

$R(u)$ is defined below (42). As for I_3 we need it only at $\delta = 0$. Then its value is $2/3$. Using (43-45) we get

$$I_0 = \frac{\pi}{2} + \delta + \frac{5}{3} \delta^3 + \frac{87}{10} \delta^5 + \dots, \quad I_1 = 1 + \frac{\pi}{2} \delta + 2\delta^2 + \dots, \quad I_2 = \frac{\pi}{4} + 2\delta + \dots \quad (50)$$

$$\frac{1}{\sqrt{u_3 \delta}} = 1 + \delta^2 + \frac{9}{2} \delta^4 + O(\delta^6).$$

Now for (42) we obtain

$$\vartheta_{1/2} = \frac{\pi}{2} + \delta + \frac{\pi}{2} \delta^2 + \frac{11}{3} \delta^3 + 3\pi \delta^4 + \frac{383}{15} \delta^5 \dots, \quad (51)$$

(We have used this method to check the equation (35)) From here

$$\delta = y \left\{ 1 - \frac{\pi}{2} y + \left(\frac{\pi^2}{2} - \frac{11}{3} \right) y^2 + \left(\frac{37\pi}{6} - \frac{5\pi^3}{8} \right) y^3 + \right.$$

$$\left. \left(\frac{74}{5} - \frac{41\pi^2}{4} + \frac{7\pi^4}{2^3} \right) y^4 + \dots \right\}, \quad y = \frac{\theta}{2} = \vartheta_{1/2} - \frac{\pi}{2}. \quad (52)$$

As in (20) we obtain δ^{-2} in the form (37) where now

$$c_1 = \pi, \quad c_2 = \frac{22}{3} - \frac{\pi^2}{4}, \quad c_3 = -\frac{4\pi}{3} + \frac{\pi^3}{4}, \quad c_4 = \frac{161}{15} + 2\pi^2 - \frac{5\pi^4}{16}. \quad (53)$$

With these c_i the equation (38) holds. Here the approximate value of c_i are

$$c_1 = 3.1416, \quad c_2 = 4.8660, \quad c_3 = 3.5627, \quad c_4 = 0.0322. \quad (53a)$$

Comparing with (37b) we see how the difference between c_i there and here increases with i .

4 Small angle classical scattering by an interval of gravitational field

This problem is usually avoided because all coordinate systems are equivalent in general relativity. Here we assume that in the privileged system the observed deflection is given by the tangent to the trajectory. There are some reason to think that the privileged system must be isotropic one [11] and we take it from general relativity.

For tangent we have

$$\tan \varphi = \frac{dy}{d\vartheta} / \frac{dx}{d\vartheta} = \frac{\frac{dr}{d\vartheta} \sin \vartheta + r \cos \vartheta}{\frac{dr}{d\vartheta} \cos \vartheta - r \sin \vartheta}, \quad x = r \cos \vartheta, \quad y = r \sin \vartheta, \quad (54)$$

or in terms of $u = \rho/r$:

$$\tan \varphi = \frac{\frac{du}{d\vartheta} \sin \vartheta - u \cos \vartheta}{\frac{du}{d\vartheta} \cos \vartheta + u \sin \vartheta}. \quad (55)$$

So we have to know $u(\vartheta)$ or $\vartheta(u)$. For a massless particle in the isotropic system we have

$$\frac{du}{d\vartheta} = \pm \sqrt{\frac{(1 + \frac{1}{4}u\delta)^6}{(1 - \frac{1}{4}u\delta)^2} - u^2}. \quad (56)$$

To get $u(\vartheta)$ from here is a much more difficult job than in the case of the standard Schwarzschild system utilized in (2).

In the following we assume for simplicity that $\delta \ll 1$ and $1 - u^2$ is of order of unity. (The latter assumption is not needed for the final result (60)) Then for the ingoing half of the trajectory we may write

$$\frac{du}{d\vartheta} = \sqrt{1 - u^2} \left(1 + \frac{u\delta}{1 - u^2} + O(\delta^2) \right). \quad (57)$$

From here we get

$$\vartheta = \arcsin u + \Delta, \quad \Delta = \left(1 - \frac{1}{\sqrt{1 - u^2}} \right) \delta. \quad (58)$$

So

$$\sin \vartheta = u + \Delta \sqrt{1 - u^2} = u + (\sqrt{1 - u^2} - 1)\delta, \quad \cos \vartheta = \sqrt{1 - u^2} - \Delta u. \quad (59)$$

Then, using (57) and (59), we find from (55)

$$\tan \varphi \approx \varphi = (1 - \sqrt{1 - u^2})\delta = \left(\frac{1}{2}u^2 + \frac{1}{8}u^4 + \dots \right) \delta \quad (60)$$

This is the contribution from the part of the trajectory beginning at $u = 0$ and ending at u .

We note here that from (59) we can obtain the trajectory in the form

$$u = \sin \theta + (1 - \cos \theta)\delta, \quad (57a)$$

which is equation (40.6) in § 40 in [5]. For this in the term with δ in the second equation in (59) we replace $\sqrt{1 - u^2}$ by its zero order value $\cos \theta$.

As is well known the leading term of the small angle classical deflection can be obtained by simple mechanical considerations, see §39, Problem 2 in [12], or equation (4.41) in Ch2 in [13]. In our case the contribution from the whole trajectory is

$$\varphi = r_g \rho \int_{-\infty}^{\infty} \frac{dx}{(x^2 + \rho^2)^{3/2}} = 2\delta. \quad (61)$$

It is assumed here that on the r.h.s. the trajectory may be taken as a straight line; $y \cos \vartheta = \rho$. The contribution from the same part of the trajectory as in (60) is ($x = r \cos \vartheta$; $y = r \sin \vartheta$)

$$r_g \rho \int_x^\infty \frac{dx}{(x^2 + \rho^2)^{3/2}} = \frac{r_g}{\rho} \left(1 - \frac{x \rho^{-1}}{\sqrt{1 + (x \rho^{-1})^2}} \right). \quad (62)$$

As

$$\frac{x}{\rho} = \frac{r}{\rho} \cos \vartheta = \frac{1}{u} \sqrt{1 - u^2}$$

the r.h.s. of (62) is equal to that of (60). Only smallness of φ is used in obtaining (62). If $u \ll 1$ then δ may be of order unity. If $\delta \ll 1$ then u may be of order unity.

On the surface of the Sun $0 \leq \varphi \leq \delta$. Zero corresponds to the radial trajectory $\rho = 0$, δ corresponds to the trajectory touching the surface. For $u \ll 1$ we have from (62) $\varphi = (r_g \rho)/(2r^2)$.

If we use the standard Schwarzschild coordinate system, see (1) and (2), we get instead of $\sin \vartheta$ in (59)

$$\sin \vartheta = u + [\sqrt{1 + u^2} + \frac{u^2}{2} - 1]\delta$$

and

$$\tan \varphi \approx \varphi = (1 - \sqrt{1 - u^2}(1 + \frac{u^2}{2}))\delta = \frac{1}{8}u^4\delta + \dots. \quad (63)$$

instead of (60). (As it should be, for $u = 1$ both (60) and (63) give $\tan \varphi = \vartheta_{1/2} = \delta$.) So the measurements of φ in the vicinity of the Sun can decide which coordinate system is the privileged one.

Finally, the contribution to φ from a finite interval of trajectory from x to \tilde{x} is

$$r_g \rho \int_x^{\tilde{x}} \frac{dx'}{(x'^2 + \rho^2)^{3/2}} = \frac{r_g}{\rho} \left[\frac{\tilde{x}}{\sqrt{\rho^2 + \tilde{x}^2}} - \frac{x}{\sqrt{\rho^2 + x^2}} \right]. \quad (64)$$

see (61). In (64) x and \tilde{x} may lie on the ingoing $x < 0$ and outgoing $\tilde{x} < 0$ halves of the trajectory respectively. In more precise approach (in terms of u) we must treat each half of the trajectory separately.

Acknowledgements

I am indebted to V.I. Ritus for stimulating discussions. The work was supported by Scientific Schools and Russian Fund for Fundamental Research (Grants 4401.2006.2 and 08-02-01118).

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